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FAST TRACK COMMUNICATION

On analytic descriptions of two-dimensional surfaces associated with the $\mathbb{C}P^{N-1}$ sigma model

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Abstract

We study analytic descriptions of conformal immersions of the Riemann sphere S^2 into the $\mathbb{C}P^{N-1}$ sigma model. In particular, an explicit expression for two-dimensional (2D) surfaces, obtained from the generalized Weierstrass formula, is given. It is also demonstrated that these surfaces coincide with those obtained from the Sym–Tafel formula. These two approaches correspond to parametrizations of one and the same surface in $\mathbb{R}^{N^2-1}$.

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In this communication, we investigate the relations between the $\mathbb{C}P^{N-1}$ sigma model and the generalized Weierstrass formula for the immersion of 2D surfaces in multi-dimensional Euclidean spaces. These links have been discussed in [1–3] and are governed by the formula

$$X_k(\xi,\bar{\xi}) = \mathrm{i}\int_\gamma (-[\partial P_k, P_k]\,\mathrm{d}\xi + [\bar{\partial} P_k, P_k]\,\mathrm{d}\bar{\xi}), \qquad k = 0, 1, \ldots, N-2, \tag{1}$$

where P_k are rank-1 orthogonal projectors which satisfy the completely integrable 2D $\mathbb{C}P^{N-1}$ sigma model:

$$\partial[\bar{\partial} P_k, P_k] + \bar{\partial}[\partial P_k, P_k] = 0, \qquad P_k^2 = P_k, \qquad P_k^\dagger = P_k. \tag{2}$$

We first demonstrate that for any solution of the $\mathbb{C}P^{N-1}$ sigma model (2) defined on the Riemann sphere S^2 with a finite action functional, the generalized Weierstrass formula for the immersion of 2D surfaces (1) can be integrated explicitly up to a constant of integration and expressed in terms of the projectors P_k:

$$X_k(\xi,\bar{\xi}) = -\mathrm{i}\left(P_k + 2\sum_{j=0}^{k-1} P_j\right), \qquad k = 0, 1, \ldots, N-2. \tag{3}$$

Indeed, if we assume that the $\mathbb{C}P^{N-1}$ sigma model is defined on the sphere S^2 with a finite action functional, then the complete set of regular solutions is known [4, 5]. As a result, one gets three classes of solutions, namely, (i) holomorphic (i.e. $\bar{\partial} f = 0$), (ii) antiholomorphic (i.e. $\partial f = 0$) and (iii) mixed. The mixed solutions can be determined from either the holomorphic or the antiholomorphic nonconstant functions by the successive application of the operator P_+ [6],

$$P_+ : f \in \mathbb{C}^N \to P_+ f = \partial f - f \frac{f^\dagger \partial f}{f^\dagger f}, \tag{4}$$

where f is any nonconstant holomorphic function. This allows one to construct mixed solutions $f_k = P_+^k f$ which represent harmonic maps from S^2 to the $\mathbb{C}P^{N-1}$ model. Here, the operator P_+^k is obtained by applying successively k times the operator P_+ and $P_+^0 = \mathrm{id}$. Hence, using (4) for every $k \leqslant N-1$ we can define a set of rank-1 projectors $\{P_0, P_1, \ldots, P_k\}$

$$P_k := \frac{f_k \otimes f_k^\dagger}{f_k^\dagger \cdot f_k}, \qquad k = 0, 1, \ldots, N-1, \tag{5}$$

which determine conservation laws of the form (2). The first ($k = 0$) and the last ($k = N-1$) conservation laws are related to the holomorphic and antiholomorphic solutions, respectively, while the intermediate ones are related to the mixed solutions. Consequently, according to the Weierstrass procedure we can obtain 2D surfaces for each projector P_k. By a straightforward calculation one gets [2]

$$\begin{aligned} [\partial P_k, P_k] &= \partial P_k + 2\frac{\left(P_+^k f\right) \otimes \left(P_+^{k-1} f\right)^\dagger}{\left|P_+^{k-1} f\right|^2}, \\ [\bar{\partial} P_k, P_k] &= -\bar{\partial} P_k - 2\frac{\left(P_+^{k-1} f\right) \otimes \left(P_+^k f\right)^\dagger}{\left|P_+^{k-1} f\right|^2}. \end{aligned} \tag{6}$$

Hence, for every $k \leqslant N-1$, the Weierstrass formula for immersion (1) takes the form

$$\mathrm{d}X_k = -\mathrm{i}\left[\left(\partial P_k + 2\frac{\left(P_+^k f\right) \otimes \left(P_+^{k-1} f\right)^\dagger}{\left|P_+^{k-1} f\right|^2}\right)\mathrm{d}\xi + \left(\bar{\partial} P_k + 2\frac{\left(P_+^{k-1} f\right) \otimes \left(P_+^k f\right)^\dagger}{\left|P_+^{k-1} f\right|^2}\right)\mathrm{d}\bar{\xi}\right]. \tag{7}$$

Note that the two surfaces corresponding to $k = 1$ and $k = N-1$ are precisely the same objects, since one gets antiholomorphic solutions of the model after applying $N-1$ times the operator P_+ to the nonconstant holomorphic function f. Hence, there appear at most $N-2$ different surfaces as a result of the generalized Weierstrass formula.

The integration of (7) gives us (3) which can be shown as follows. It is immediately seen that for $k = 0$ we have $X_0 = -\mathrm{i}P_0$ and upon differentiation we obtain

$$\mathrm{d}X_0 = -\mathrm{i}[\partial P_0 \, \mathrm{d}\xi + \bar{\partial} P_0 \, \mathrm{d}\bar{\xi}], \tag{8}$$

which coincides with (7) for $k = 0$. For $k = 1$, we need to show that

$$\partial P_0 = \frac{(P_+ f) \otimes f^\dagger}{|f|^2}, \tag{9}$$

which could easily be computed by differentiating P_0 and bearing in mind that f is holomorphic. In order to show that (3) holds for any k we assume that

$$\partial(P_0 + P_1 + \cdots, P_{k-2}) = \frac{\left(P_+^{k-1} f\right) \otimes \left(P_+^{k-2} f\right)^\dagger}{\left|P_+^{k-2} f\right|^2}, \tag{10}$$

and then compute ∂P_{k-1}

$$\partial P_{k-1} = \partial \left[\frac{\left(P_+^{k-1} f\right) \otimes \left(P_+^{k-1} f\right)^\dagger}{\left|P_+^{k-1} f\right|^2} \right]$$
$$= \frac{\left(P_+^{k} f\right) \otimes \left(P_+^{k-1} f\right)^\dagger}{\left|P_+^{k-1} f\right|^2} - \frac{\left(P_+^{k-1} f\right) \otimes \left(P_+^{k-2} f\right)^\dagger}{\left|P_+^{k-2} f\right|^2}, \tag{11}$$

where we have used the fact that

$$\partial \left(P_+^{k-1} f\right)^\dagger = -\frac{\left|P_+^{k-1} f\right|^2}{\left|P_+^{k-2} f\right|^2} \left(P_+^{k-2} f\right)^\dagger, \tag{12}$$

together with the orthogonality relation

$$\left(P_+^{k} f\right)^\dagger \cdot \left(P_+^{l} f\right) = 0, \qquad \text{for} \quad k \neq l. \tag{13}$$

Thus, we have shown that

$$\partial \left(\sum_{j=0}^{k-1} P_j \right) = \frac{\left(P_+^{k} f\right) \otimes \left(P_+^{k-1} f\right)^\dagger}{\left|P_+^{k-1} f\right|^2}, \tag{14}$$

which indeed justifies (3).

It is interesting to note that this immersion function X_k coincides with the results obtained in [2], namely the surface for nonholomorphic Veronese-type solutions lives in $\mathbb{R}^3$.

For the $\mathbb{C}P^{N-1}$ sigma model, it may also be of interest to investigate the links between the generalized Weierstrass formula for the immersion of 2D surfaces in the $su(N)$ algebra and the approach based on the linear spectral problem for constructing infinitely many surfaces in multi-dimensional Euclidean spaces $\mathbb{R}^{N^2-1} \simeq su(N)$.

Following the approach proposed by Sym and Tafel in [7–10], in particular using their formula

$$X_k(\xi, \bar{\xi}) = \alpha(\lambda) \phi_k^{-1} \frac{\partial \phi_k}{\partial \lambda}, \qquad \phi_k \in SU(N), \qquad k = 0, 1, \ldots, N-2, \tag{15}$$

for integrable surfaces derived via the Lax pair [11, 12]

$$\partial \phi_k = \frac{2}{1+\lambda} [\partial P_k, P_k] \phi_k, \qquad \bar{\partial} \phi_k = \frac{2}{1-\lambda} [\bar{\partial} P_k, P_k] \phi_k, \qquad k = 0, 1, \ldots, N-1, \tag{16}$$

we demonstrate that there exist 2D surfaces with $su(N)$-valued immersion functions $X_k(\xi, \bar{\xi})$ which are precisely of the form (3). Here, λ is a spectral parameter and the compatibility conditions for the system (16) coincide with the $\mathbb{C}P^{N-1}$ model equations (2). Furthermore, $\alpha(\lambda)$ is some scalar function of λ.

In the purely instantonic case (i.e. holomorphic and antiholomorphic solutions), the orthogonal projector P has the form

$$P_0 = \frac{f \otimes f^\dagger}{f^\dagger \cdot f}, \tag{17}$$

which satisfies

$$[\partial P_0, P_0] = \frac{P_+ f \otimes f^\dagger}{f^\dagger \cdot f}, \qquad [\bar{\partial} P_0, P_0] = -\frac{f \otimes (P_+ f)^\dagger}{f^\dagger \cdot f}. \tag{18}$$

Looking for a solution $\phi_0 = \phi_0(\lambda)$ of the linear spectral problem (16) when ϕ_0 tends to 1 as $\lambda \to \infty$, we make the Ansatz [6],

$$\phi_0 = I_N - \frac{2}{1-\lambda} P_0, \tag{19}$$

where I_N is the $N \times N$ identity matrix. The inverse matrix of ϕ_0 is given by

$$\phi_0^{-1} = I_N - \frac{2}{1+\lambda} P_0. \tag{20}$$

Hence, according to the Sym–Tafel formula (15), the surface associated with the $\mathbb{C}P^{N-1}$ model is given up to an additive $su(N)$-valued constant by

$$X_0 = \frac{2}{1-\lambda^2} P_0. \tag{21}$$

For the nonholomorphic solutions (i.e. the mixed solutions), we proceed in an analogous way and find that [6]

$$\phi_k = I_N + \frac{4\lambda}{(1-\lambda)^2} \sum_{j=0}^{k-1} P_j - \frac{2}{1-\lambda} P_k, \qquad k = 1, \ldots, N-2, \tag{22}$$

and the inverse of ϕ_k has the form

$$\phi_k^{-1} = I_N - \frac{4\lambda}{(1+\lambda)^2} \sum_{j=0}^{k-1} P_j - \frac{2}{1+\lambda} P_k. \tag{23}$$

The Sym–Tafel formula (15) for the immersion function $X_k(\xi, \bar{\xi})$ of 2D surfaces associated with the $\mathbb{C}P^{N-1}$ model is given up to an additive $su(N)$-valued constant by the formula

$$X_k(\xi, \bar{\xi}) = \frac{2}{1-\lambda^2} \left(P_k + 2 \sum_{j=0}^{k-1} P_j \right), \qquad k = 0, 1, \ldots, N-2. \tag{24}$$

This result for the immersion of 2D surfaces in the $su(N)$ Lie algebras coincides with that obtained from the Weierstrass representation (3) and shows the equivalence between these two approaches.

To conclude, we give an explicit expression for 2D surfaces, obtained from the generalized Weierstrass formula, and demonstrate that these surfaces coincide with those obtained from the Sym–Tafel formula.

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